

9 部分空間の基底と次元

問題 9.1. (1) ならない. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in W$ だが $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin W$.

(2) なる. $W = \left\{ \mathbf{x} = \begin{pmatrix} a \\ a \\ b \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}}$.

(3) なる. $W = \left\{ \mathbf{x} = \begin{pmatrix} 3a \\ 4a \\ -2a \end{pmatrix} \middle| a \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \right\rangle_{\mathbb{R}}$.

(4) ならない. $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \in W$ だが $0 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin W$.

(5) なる. $W = \left\{ \mathbf{x} = \begin{pmatrix} 0 \\ a \\ -a \end{pmatrix} \middle| a \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle_{\mathbb{R}}$.

(6) ならない. $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in W$ だが $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \notin W$.

問題 9.2.

(1) $N(A) = \left\langle \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}}$, 基底: $\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}$.

(2) $N(A) = \left\langle \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}}$, 基底: $\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$.

(3) $N(A) = \left\langle \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}}$, 基底: $\left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\}$.

(4) $N(A) = \left\langle \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}}$, 基底: $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(5) $N(A) = \left\langle \begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}}$, 基底: $\left\{ \begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(6) $N(A) = \left\langle \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right\rangle_{\mathbb{R}}$, 基底: $\left\{ \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right\}$.

問題 9.3.

(1) 基底: $\left\{ \begin{pmatrix} 10 \\ 3 \\ 1 \end{pmatrix} \right\}$, $\dim W = 1$.

(2) 基底: $\left\{ \begin{pmatrix} -14 \\ -11 \\ 1 \end{pmatrix} \right\}$, $\dim W = 1$.

(3) 基底: $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\}$, $\dim W = 1$.

(4) 基底: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, $\dim W = 1$.

(5) 基底: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$, $\dim W = 2$.

(6) 基底: $\left\{ \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}, \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}$, $\dim W = 2$.

問題 9.4. $x \in W$, $y \in W$ のとき $x + y \in W$, $ax \in W$ を示せばよい.

$$(1) A(x + y) = Ax + Ay = \mathbf{0} + \mathbf{0} = \mathbf{0}, A(ax) = a(Ax) = a\mathbf{0} = \mathbf{0}.$$

(2) $y_1, y_2 \in W$ とすると $y_1 = x_{11} + x_{12}$, $Ax_{11} = \mathbf{0}$, $Bx_{12} = \mathbf{0}$, $y_2 = x_{21} + x_{22}$, $Ax_{21} = \mathbf{0}$, $Bx_{22} = \mathbf{0}$ となる. $y_1 + y_2 = (x_{11} + x_{21}) + (x_{12} + x_{22})$. $A(x_{11} + x_{21}) = Ax_{11} + Ax_{21} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, $B(x_{12} + x_{22}) = Bx_{12} + Bx_{22} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ なので $y_1 + y_2 \in W$. $ay_1 = ax_{11} + ax_{12}$, $A(ax_{11}) = a(Ax_{11}) = a\mathbf{0} = \mathbf{0}$, $B(ax_{12}) = a(Bx_{12}) = a\mathbf{0} = \mathbf{0}$ なので $ay_1 \in W$.

(3) $y_1, y_2 \in W$ とすると $y_1 = Ax_{11} + Bx_{12}$, $y_2 = Ax_{21} + Bx_{22}$ となる. $y_1 + y_2 = (Ax_{11} + Bx_{12}) + (Ax_{21} + Bx_{22}) = A(x_{11} + x_{21}) + B(x_{12} + x_{22})$. なので $y_1 + y_2 \in W$. $ay_1 = a(Ax_{11} + Bx_{12}) = A(ax_{11}) + B(ax_{12})$ なので $ay_1 \in W$.

問題 9.5.

$$(1) \text{ 基底 : } \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ -8 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 5 \end{pmatrix} \right\}, \dim W = 2. \quad (2) \text{ 基底 : } \left\{ \begin{pmatrix} 1 \\ 0 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 7 \\ -4 \end{pmatrix} \right\}, \dim W = 2.$$

$$(3) \text{ 基底 : } \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 14 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 9 \end{pmatrix} \right\}, \dim W = 2. \quad (4) \text{ 基底 : } \left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix} \right\}, \dim W = 2.$$

$$(5) \text{ 基底 : } \left\{ \begin{pmatrix} 1 \\ 3 \\ -8 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 15 \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}, \dim W = 2.$$

$$(6) \text{ 基底 : } \left\{ \begin{pmatrix} 5 \\ 2 \\ -6 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -18 \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix} \right\}, \dim W = 2.$$

$$(7) \text{ 基底 : } \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}, \dim W = 2.$$

$$(8) \text{ 基底 : } \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}, \dim W = 2.$$

$$(9) \text{ 基底 : } \left\{ \begin{pmatrix} 3 \\ 2 \\ 13 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 12 \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}, \dim W = 2.$$

$$(10) \text{ 基底 : } \left\{ \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}, \dim W = 2.$$