13　対称行列の対角化

問題 13.1. 対角化に必要な直交行列 P や出来上がる対角行列は異なるものもあり得る.

(1) $P = \dfrac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}$.

(2) $P = \dfrac{1}{\sqrt{10}}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix}$.

(3) $P = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$.

(4) $P = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$.

(5) $P = \dfrac{1}{\sqrt{10}}\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}$.

(6) $P = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}$.

(7) $P = \dfrac{1}{\sqrt{13}}\begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 10 & 0 \\ 0 & -3 \end{pmatrix}$.

(8) $P = \dfrac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 20 & 0 \\ 0 & -5 \end{pmatrix}$.

問題 13.2. 対角化に必要な直交行列 P や出来上がる対角行列は異なるものもあり得る.

(1) $P = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} = \dfrac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & \sqrt{3} \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(2) $P = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(3) $P = \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} = \dfrac{1}{\sqrt{6}}\begin{pmatrix} 2 & -\sqrt{2} & 0 \\ 1 & \sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{2} & \sqrt{3} \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -5 \end{pmatrix}$.

(4) $P = \begin{pmatrix} 1/3 & -2/\sqrt{5} & -2/3\sqrt{5} \\ 2/3 & 1/\sqrt{5} & -4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{pmatrix} = \dfrac{1}{3\sqrt{5}}\begin{pmatrix} \sqrt{5} & -6 & -2 \\ 2\sqrt{5} & 3 & -4 \\ 2\sqrt{5} & 0 & 5 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$.

(5) $P = \begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} = \dfrac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \\ \sqrt{2} & \sqrt{3} & 1 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

(6) $P = \begin{pmatrix} 3/\sqrt{11} & -1/\sqrt{10} & -3/\sqrt{110} \\ 1/\sqrt{11} & 3/\sqrt{10} & -1/\sqrt{110} \\ 1/\sqrt{11} & 0 & 10/\sqrt{110} \end{pmatrix} = \dfrac{1}{\sqrt{110}}\begin{pmatrix} 3\sqrt{10} & -\sqrt{11} & -3 \\ \sqrt{10} & 3\sqrt{11} & -1 \\ \sqrt{10} & 0 & 10 \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} -10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(7) $P = \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} = \dfrac{1}{\sqrt{6}}\begin{pmatrix} 2 & -\sqrt{2} & 0 \\ -1 & -\sqrt{2} & \sqrt{3} \\ 1 & \sqrt{2} & \sqrt{3} \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(8) $P = \begin{pmatrix} -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix} = \dfrac{1}{\sqrt{6}}\begin{pmatrix} -1 & -\sqrt{3} & \sqrt{2} \\ -1 & \sqrt{3} & \sqrt{2} \\ 2 & 0 & \sqrt{2} \end{pmatrix}$, ${}^tPAP = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

問題 13.3.

$$R(-\theta)AR(\theta) = \frac{1}{2}\begin{pmatrix} 2hsin2\theta + (a-b)\cos 2\theta + a + b & (b-a)\sin 2\theta + 2h\cos 2\theta \\ (b-a)\sin 2\theta + 2h\cos 2\theta & -2h\sin 2\theta - (a-b)\cos 2\theta + a + b \end{pmatrix}$$

だから $(b-a)\sin 2\theta + 2h\cos 2\theta = 0$ となる θ であれば対角行列になる.

問題 13.4.

(1) $\theta = \dfrac{\pi}{6},\ R(-\theta)AR(\theta) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$ (2) $\theta = \dfrac{\pi}{4},\ R(-\theta)AR(\theta) = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}.$

(3) $\theta = \dfrac{\pi}{3},\ R(-\theta)AR(\theta) = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}.$ (4) $\theta = \dfrac{\pi}{4},\ R(-\theta)AR(\theta) = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}.$

問題 13.5.

$$(1)\ P = \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{15} \\ 2/\sqrt{10} & 0 & -1/\sqrt{3} & -2/\sqrt{15} \\ 1/\sqrt{10} & -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{15} \\ 2/\sqrt{10} & 0 & 0 & 3/\sqrt{15} \end{pmatrix} = \frac{1}{\sqrt{30}} \begin{pmatrix} \sqrt{3} & \sqrt{15} & \sqrt{10} & -\sqrt{2} \\ 2\sqrt{3} & 0 & -\sqrt{10} & -2\sqrt{2} \\ \sqrt{3} & -\sqrt{15} & \sqrt{10} & -\sqrt{2} \\ 2\sqrt{3} & 0 & 0 & 3\sqrt{2} \end{pmatrix},$$

$${}^tPAP = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

$$(2)\ P = \begin{pmatrix} 1/2 & -1/\sqrt{2} & 1/\sqrt{6} & 1/2\sqrt{3} \\ 1/2 & 1/\sqrt{2} & 1/\sqrt{6} & 1/2\sqrt{3} \\ -1/2 & 0 & 2/\sqrt{6} & -1/2\sqrt{3} \\ -1/2 & 0 & 0 & 3/2\sqrt{3} \end{pmatrix} = \frac{1}{2\sqrt{6}} \begin{pmatrix} \sqrt{6} & -2\sqrt{3} & 2 & \sqrt{2} \\ \sqrt{6} & 2\sqrt{3} & 2 & \sqrt{2} \\ -\sqrt{6} & 0 & 4 & -\sqrt{2} \\ -\sqrt{6} & 0 & 0 & 3\sqrt{2} \end{pmatrix},$$

$${}^tPAP = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

$$(3)\ P = \begin{pmatrix} 1/\sqrt{2} & -2/\sqrt{10} & -1/3\sqrt{2} & 2/3\sqrt{10} \\ 0 & -1/\sqrt{10} & 4/3\sqrt{2} & 1/3\sqrt{10} \\ 1/\sqrt{2} & 2/\sqrt{10} & 1/3\sqrt{2} & -2/3\sqrt{10} \\ 0 & 1/\sqrt{10} & 0 & 9/3\sqrt{10} \end{pmatrix} = \frac{1}{3\sqrt{10}} \begin{pmatrix} 3\sqrt{5} & -6 & -\sqrt{5} & 2 \\ 0 & -3 & 4\sqrt{5} & 1 \\ 3\sqrt{5} & 6 & \sqrt{5} & -2 \\ 0 & 3 & 0 & 9 \end{pmatrix},$$

$${}^tPAP = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix}.$$